Two-particle Wigner functions in a one-dimensional Calogero-Sutherland potential

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Abstract

We calculate the Wigner distribution function for the Calogero-Sutherland system which consists of harmonic and inverse-square interactions. The Wigner distribution function is separated out into two parts corresponding to the relative and center-of-mass motions. A general expression for the relative Wigner function is obtained in terms of the Laguerre polynomials by introducing a new identity between Hermite and Laguerre polynomials.

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1 Introduction

Since the introduction of the Wigner function (WF) in 1932 for inclusion of quantum corrections to classical results[1], phase space representations of the quantum mechanics have been a focus of continuing interest and found a wide range of applications[2]. In this formalism one defines a distribution function $W(q,p)$ of position q and momentum p in such a way that to every normalized state vector ψ there corresponds a distribution function. In order to define the same physical system, W should be a Hermitian form of ψ , and if W is integrated over p it should give the proper probabilities of the different values of q or vice versa. Since the WF is a probability distribution function one expects, as a natural condition on $W(q,p)$, that it should be non-negative for all values of q and p: $W(q,p) \geq 0$. However ,Wigner[3] proved that this result is incompatible with the first two conditions and it is now a common practice to work with a distribution function taking negative values for certain q and p in the phase space.

The quantum Calogero-Sutherland model (CSM) having a quadratic confining q^2 plus an inversely quadratic $1/q^2$ potentials[4] has applications in a wide variety of different areas of many body physics, due to the connection of its variants and itself directly with the hierarchical fractional quantum Hall effect[5], free oscillators on a circle[6], the spectrum of the Chern-Simons matrix model[7], short range Dyson model[8], and Witten-Dijkgraff-Verlinde equation[9]. Additionally, many works have also been realized to construct its N-fermion version[10], W_∞ algebra unification[11], shape invariance[12], generalized statistics[13], statistical properties of quantum quasi-degenaracy[14], equivalence to decoupled oscillators[15].

In addition to allowing one to analyze the dynamics of quantum systems entirely in phase space and thus to make comparison between their classical and quantum evolutions, there is also experimental interest on the measurements of WFs for certain quantum systems to probe the predictions of quantum mechanics, since WF contains complete quantum mechanical information as the wave function or density matrix has. A number of experiments have been reported where the measurement of WFs carried out for both vacuum and quadrature-squeezed states of light[16], molecular vibrational states[17], various quantum states of the motion of a harmonically trapped atom [18], as well as for a massive particle wave packet[19]. Of particular importance are the experiments upon which the negatives in WFs corresponding to Fock states[18] and a superposition of macroscopically separated parts of matter field[19] have been observed. In this regard, due to the fact that it is now possible to realize new quantum mesoscopic devices such as quantum dots and quantum antidots by various experimental techniques, the CSM may serve as a model of two non-interacting electrons with an individual quantum antidot confined in a quantum wire or a stripe, where the repulsive inverse-square quantum antidot potential acts as a scattering center for electrons, or may be used as a one dimensional exactly soluble band model or quantum dot arrays wherein the coupling constant of the inverse-square potential of the CSM is chosen as $0 \geq g \geq -1/2$ in dimensionless units, i.e., attractive[20, 21]. Moreover, very recently, Li *et al.*[22] have solved CSM with pseudo-angular momentum operator method, and they showed that, by discussing its several variants, the radical equations of three dimensional isotropic oscillator and hydrogen-like atom in both spherical and parabolic coordinates, one dimensional three body problem and the s-state of Morse potential all reduces to CSM. Therefore, WFs of CSM may provide a solid basis for the discussions of transport properties[23] of the above mentioned nanostructures. With these motivations, we study the WFs of CSM which has not only a particular significance in itself, but also enables us to understand the phase space picture of its variants and itself as well.

The WF for two particles is defined by

$$\begin{aligned}
&W\left(q_1, q_2; p_1, p_2\right) = \frac{1}{\left(\pi\hbar\right)^2} \int_{-\infty}^{+\infty} \mathrm{dy}_1 \int_{-\infty}^{+\infty} \mathrm{dy}_2 \\
&\times \bar{\Psi}^* \left(q_1 + y_1, q_2 + y_2\right) \bar{\Psi} \left(q_1 - y_1, q_2 - y_2\right) \\
&\times \exp\left[2i\left(p_1 y_1 + p_2 y_2\right)/\hbar\right]. \qquad (1)
\end{aligned}$$

If we express the WF in terms of the center-of-mass and relative coordinates through the relations

$$\begin{aligned}
q_1 + q_2 &= 2Q\,, \qquad q_1 - q_2 = q \\
p_1 + p_2 &= 2P\,, \qquad p_1 - p_2 = p \\
y_1 + y_2 &= 2Y\,, \qquad y_1 - y_2 = y
\end{aligned}$$

then, provided that the interparticle potential depends on the relative coordinates, equation (1) becomes

$$\begin{aligned}
&W\left(q, Q; p, P\right) = \frac{1}{\left(\pi\hbar\right)^2} \int_{-\infty}^{+\infty} \mathrm{dy} \int_{-\infty}^{+\infty} \mathrm{dY} \\
&\times \Psi^* \left(q + y\right) \Psi^* \left(Q + Y\right) \Psi \left(q - y\right) \Psi(Q - Y) \\
&\times \exp\left[i\left(4PY + py\right)/\hbar\right], \qquad (2)
\end{aligned}$$

since the solution of the corresponding Schrödinger equation can be represented by a product of two functions, one for the center-of-mass and other for the relative coordinates. Then, Eq. (2) can be

separated as and relative WFs, $W(q,p)$ and $W(Q,P)$, respectively. We can therefore define the WFs for the center-of-mass and relative motions in the form

$$W(Q,P) = \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{dY}\ \Psi^*(\mathrm{Q}+\mathrm{Y})\ \ \Psi(\mathrm{Q}-\mathrm{Y})\exp\left[4\mathrm{iPY}/\hbar\right], \tag{3}$$

$$W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{dy}\ \Psi^*(\mathrm{q}+\mathrm{y})\ \ \Psi(\mathrm{q}-\mathrm{y})\exp\left[\mathrm{ipy}/\hbar\right], \tag{4}$$

respectively. It should be noted that, while $\bar{\Psi}$'s in Eq. (1) represent the two body wavefunctions, Ψ's in Eq. (2-4) are single particle wave functions. Now, it is possible to present a coupled system of linear partial differential equations corresponding to the above defined WF, which requires the direct computation without solving the wave functions[24]. The Wigner representation is very convenient for studying quantum systems with Hamiltonians that include quadratic coordinates and momenta, since in this case the Wigner distribution function represents a good approximate description of the dynamics involved. The method, however, is not easy to handle when the potential contains higher order powers of coordinates, since this case comprises a differential equation for the Wigner function with terms as much as the number of the order . In recent years there has been a number of works to calculate the WF for various type of potentials: Infinite square well[25], a double well potential[26], the Pösch-Teller potential[27], the Morse oscillator[28], a quantum damped oscillator[29], the hydrogen atom[30], the rotational motion of a spherical top[31] are notable applications. A discrete WF for non-relativistic quantum systems with one degree of freedom has been developed in finite dimensional phase space and applied to a few simple system[32].

The layout of this paper is as follows: In Sec. 2, we discuss WFs for the center-of-mass and relative motions, and obtain general expressions in terms of Laguerre polynomials. In Sec. 3, we derive a new identity between Hermite and Laguerre polynomials to obtain a compact form for the WF of the relative part, and plot some of them for a few states to give an idea on their phase space behaviors.

2 Theory

The Hamiltonian describing two particles interacting pairwise by the Calogero-Sutherland potential is given by

$$H = \sum_{i=1}^{2}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_i^2} + \frac{1}{2}m\omega_\bullet^2 q_i^2 + \frac{1}{2}\sum_{j\neq i}^{2} U\left(|q_i - q_j|\right)\right], \tag{5}$$

where the second term is the confining potential and

$$U\left(|q_i - q_j|\right) = \left[m\omega_0^2\left(q_i - q_j\right)^2 + 2g/\left(q_i - q_j\right)^2\right]/2$$

simulates further interactions between two particles. If we now use the above defined center-of-mass and relative coordinates, then the relevant Schrödinger equation is separated out as a center-of-mass equation, which is a 1D harmonic oscillator equation

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dQ^2} + \frac{1}{2}M\omega_\bullet^2 Q^2\right]\Psi(Q) = E_Q\Psi(Q), \tag{6}$$

and the Calogero-Sutherland system

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dq^2}+\frac{1}{2}\mu\omega^2q^2+\frac{g}{q^2}\right]\Psi\left(q\right)=E_q\Psi\left(q\right), \tag{7}$$

where we have defined $\omega^2=\omega_\bullet^2+2\omega_0^2$ as a hybrid frequency. In Eqs. (6)-(7), M and μ are total and reduced masses, and are given by $2m$ and $m/2$, respectively.

The Wigner function corresponding to the center-of-mass motion defined by Eq. (3) through the solution of Eq. (6) is well-known, and given by[2]

$$\begin{aligned} W_\ell\left(Q,\widetilde{P}\right) &= \frac{\left(-1\right)^\ell}{\pi\hbar}\exp\left[-M\omega_\bullet Q^2/\hbar-\widetilde{P}^2/M\omega_\bullet\hbar\right] \\ &\quad\times L_\ell\left(\frac{2M\omega_\bullet}{\hbar}Q^2+\frac{2\widetilde{P}^2}{\hbar M\omega_\bullet}\right), \end{aligned} \tag{8}$$

where $\widetilde{P}=2P$ is used for the sake of comparison with the results presented in the associated literature, and ℓ takes values $0,1,2,\ldots$. Although it is possible to obtain the WFs for the simple harmonic oscillator (Eq. (8)) in various ways, for example, by using algebraic methods or by solving ordinary differential equations of WF [33], the WF corresponding to the relative motion resulting in the CS system cannot be obtained by either methods. Therefore, we are compelled to obtain the corresponding Wigner function through solving Eq. (7). First, we make a change of variable by $z=\left(\mu\omega/\hbar\right)^{1/2}q$, which transforms Eq. (7) into

$$\Psi''+\left(4n+2\beta+2-z^2+\frac{1/4-\beta^2}{z^2}\right)\Psi=0, \tag{9}$$

where the new parameters are given by

$$E_q=\hbar\omega\left(2n+\beta+1\right)\ ,\ 1/4-\beta^2=-2\mu g/\hbar^2$$

with $n=0,1,2,\ldots$ and $g\geq-\hbar^2/8\mu$. It should be noted that these are the energy levels of one-dimensional isotropic harmonic oscillator with odd quantum numbers shifted by an amount $\left(\beta-1/2\right)\hbar\omega$. The solution to Eq. (9) can then be written[34] in terms of the Laguerre polynomials

$$\Psi_n\left(q\right)=C_n\ b^{\alpha/2}q^\alpha\exp\left[-bq^2/2\right]\ L_n^{\alpha-1/2}\left(bq^2\right), \tag{10}$$

where $\alpha=\beta+1/2$, $b=\mu\omega/\hbar$ and the normalization constant is given by $C_n=b^{1/4}\left[n!/\Gamma\left(n+\alpha+1/2\right)\right]^{1/2}$. We can now build the associated WFs for the relative motion (RM) with the wave functions given by Eq. (10) according to the definition of Eq. (4), which results in

$$\begin{aligned} W_{n\alpha}\left(q,\widetilde{p}\right) &= \frac{\left|C_n\right|^2}{\pi\hbar}\ b^\alpha\exp\left[-bq^2\right]\int_{-\infty}^{+\infty}\mathrm{dy}\ \left(\mathrm{q}^2-\mathrm{y}^2\right)^\alpha \\ &\quad\times\exp\left[-by^2\right]\ L_n^{\alpha-\frac{1}{2}}\left[b\left(q+y\right)^2\right] \\ &\quad\times L_n^{\alpha-\frac{1}{2}}\left[b\left(q-y\right)^2\right]\ \exp\left(2i\widetilde{p}y/\hbar\right), \end{aligned} \tag{11}$$

where $p=2\ \widetilde{p}$ is used. If we use the binomial expansion of

$$\left(a+d\right)^\alpha=\sum_{\beta=0}^{\alpha}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)a^{\alpha-\beta}\ d^\beta,$$

and series expansion of the Laguerre polynomials

$$L_n^k(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{pmatrix} k+n \\ n-m \end{pmatrix} x^m$$

then Eq. (11) is expressed in the form

$$\begin{aligned} W_{n\alpha}(q,\widetilde{p}) &= \frac{|C_n|^2}{\pi\hbar} b^\alpha \exp\left[-bq^2\right] \\ &\times \sum_m^n \sum_r^n \frac{(-1)^{m+r}}{m!r!} \begin{pmatrix} \alpha-\frac{1}{2}+n \\ n-m \end{pmatrix} \begin{pmatrix} \alpha-\frac{1}{2}+n \\ n-r \end{pmatrix} b^{m+r} \\ &\times \sum_{\beta=0}^{\alpha} (-1)^\beta \begin{pmatrix} \alpha\beta \end{pmatrix} q^{2\alpha-2\beta} \sum_{\mu=0}^{2m} \begin{pmatrix} 2m \\ \mu \end{pmatrix} q^{2m-\mu} \\ &\times \sum_{\rho=o}^{2r} (-1)^\rho \begin{pmatrix} 2r \\ \rho \end{pmatrix} q^{2r-\rho}\ F_{2\beta+\mu+\rho}(\widetilde{p}), \end{aligned} \tag{12}$$

where $F_{2\beta+\mu+\rho}(\widetilde{p})$ is given by the following integral

$$F_{2\beta+\mu+\rho}(\widetilde{p}) = \int_{-\infty}^{+\infty} dy\ y^{2\beta+\mu+\rho} \exp\left[-by^2 + 2i\widetilde{p}y/\hbar\right]. \tag{13}$$

It is easy to show that this last integral can be expressed in terms of the Hermite polynomial as follows:

$$F_n(\widetilde{p}) = b^{-(n+1)/2} \frac{\sqrt{\pi}}{2^n(-i)^n} \exp\left[-\widetilde{p}^2/b\hbar^2\right]\ H_n\left(\frac{\widetilde{p}}{\hbar\sqrt{b}}\right), \tag{14}$$

with $n = 2\beta+\mu+\rho$[35]. Furthermore, the use of the relation[35] $H_n(u) = (-1)^n\ e^{u^2}\partial_u^n\ e^{-u^2}$ for the Hermite polynomials reduces the above expression to

$$F_n(\widetilde{p}) = \sqrt{\frac{\pi}{b}} \left(-\frac{i\hbar}{2}\,\partial_{\widetilde{p}}\right)^n\ e^{-\widetilde{p}^2/b\hbar^2}, \tag{15}$$

where $\partial_{\widetilde{p}}$ represents the differentiation with respect to $\widetilde{p}$. Hence, the WFs for the RM becomes, with these new definitions,

$$\begin{aligned} W_{n\alpha}(q,\widetilde{p}) &= \frac{|C_n|^2}{\sqrt{\pi b}\hbar}\ \exp\left[-bq^2\right] \\ &\times \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{pmatrix} \alpha-\frac{1}{2}+n \\ n-m \end{pmatrix} b^m\ \left(q - \frac{i\hbar}{2}\,\partial_{\widetilde{p}}\right)^{2m} \\ &\times \sum_{r=0}^{n} \frac{(-1)^n}{n!} \begin{pmatrix} \alpha-\frac{1}{2}+n \\ n-r \end{pmatrix} b^r\ \left(q + \frac{i\hbar}{2}\,\partial_{\widetilde{p}}\right)^{2r} \\ &\times b^\alpha \left(q^2 + \frac{\hbar^2}{4}\partial_{\widetilde{p}}^2\right)^\alpha\ e^{-\widetilde{p}^2/b\hbar^2}. \end{aligned} \tag{16}$$

By using once again series expansion of the Laguerre polynomials, it is possible to express Eq. (16) in an implicit form as well

$$\begin{aligned} W_{n\alpha}(q,p) &= \frac{n!}{\sqrt{\pi}\hbar\Gamma(n+\alpha+1/2)} e^{-\mu\omega q^2/\hbar} \\ &\times L_n^{\alpha-1/2}\left[\frac{\mu\omega}{\hbar}(q-i\hbar\,\partial_p)^2\right] L_n^{\alpha-1/2}\left[\frac{\mu\omega}{\hbar}(q+i\hbar\,\partial_p)^2\right] \\ &\times \left[\frac{\mu\omega}{\hbar}\left(q^2+\hbar^2\partial_p^2\right)\right]^{\alpha} e^{-p^2/4\mu\omega\hbar}. \end{aligned} \tag{17}$$

If we now define a unit of dimension by $l=\sqrt{\hbar/m\omega_\bullet}$ with $\mu=m/2$ and $M=2m$, then we can make positions and momenta dimensionless by $\overline{Q}=Q/l$, $\overline{P}=lP/\hbar$, $\overline{q}=q/l$, $\overline{p}=lp/\hbar$, and frequency by $\overline{\omega}=\omega/\omega_\bullet$. Hence, the relevant WFs for the center-of-mass and relative motions, Eqs. (8) and (17), become

$$\widetilde{W}_\ell\left(\overline{Q},\overline{P}\right)=(-1)^\ell \exp\left[-2\overline{Q}^2-2\overline{P}^2\right]\, L_\ell\left(4\overline{Q}^2+4\overline{P}^2\right), \tag{18}$$

and

$$\begin{aligned} \widetilde{W}_{n\alpha}(\overline{q},\overline{p}) &= \frac{\sqrt{\pi}n!}{\Gamma(n+\alpha+1/2)} e^{-\overline{\omega}\overline{q}^2/2} \\ &\times L_n^{\alpha-1/2}\left[\frac{\overline{\omega}}{2}(\overline{q}-i\,\partial_{\overline{p}})^2\right] L_n^{\alpha-1/2}\left[\frac{\overline{\omega}}{2}(\overline{q}+i\,\partial_{\overline{p}})^2\right] \\ &\times \left[\frac{\overline{\omega}}{2}\left(\overline{q}^2+\partial_{\overline{p}}^2\right)\right]^{\alpha} e^{-\overline{p}^2/2\overline{\omega}}, \end{aligned} \tag{19}$$

respectively, where we have denoted $\pi\hbar W_\ell\left(\overline{Q},\overline{P}\right)$ and $\pi\hbar W_{n\alpha}(\overline{q},\overline{p})$ as $\widetilde{W}_\ell\left(\overline{Q},\overline{P}\right)$ and $\widetilde{W}_{n\alpha}(\overline{q},\overline{p})$, respectively.

3 Results and Discussion

FIGs. 1 and 2 show, respectively, the WFs $\widetilde{W}_{02}$ and $\widetilde{W}_{03}$ for the relative motion given by Eq. (19) as functions of dimensionless position $\overline{q}=q/l$ and momentum $\overline{p}=lp/\hbar$ for two different dimensionless frequency values, $\overline{\omega}=\omega/\omega_\bullet=1$ and 3. Contour plots showing the projections of the relevant WF onto $(\overline{q},\overline{p})$ plane are also shown in these figures. In other words, each contour is a slice of given WF in the $(\overline{q},\overline{p})$ plane. It should be noted that, while $\overline{\omega}=1$ corresponds to the case $\overline{\omega}_0=0$, $\overline{\omega}=3$ corresponds to switch on $\overline{\omega}_0$ to the value $\overline{\omega}_0=1$, which causes localization in $\overline{q}$[36, 37]. In addition to this pattern, delocalization in $\overline{p}$ is observed in both figures. In other words, in FIG. 1(b) and FIG. 2(b), there the dips and peaks of WFs in $\overline{q}$ are shifted towards the smaller $\overline{q}$ values compared with those in FIG. 1(a) and FIG. 2(a), whereas those of WFs in $\overline{p}$ are shifted towards higher $\overline{p}$ values.

Having obtained a general expression for the WF of two interacting particles we now distinguish between the cases $g=0$ and $g\neq 0$, and deal with each case separately. This allows us to verify the consistency of the WFs obtained above with those found in the literature. In order to see this, we need to set $g=0$ first. In this case, we have only the solutions with $\beta=+1/2$ and $-1/2$ corresponding to $\alpha=1$ and 0, respectively. We obtain, for $n=0,1,2,3,\ldots$,

$$\widetilde{W}_{n1}(\overline{q},\overline{p}) \;=\; -\frac{\sqrt{\pi}\,(2n+1)!}{n!\;2^{2n+1}\;\Gamma(n+3/2)}\exp\left(-\frac{\overline{\omega}\overline{q}^2}{2}-\frac{\overline{p}^2}{2\overline{\omega}}\right)\; L_{2n+1}\left(\overline{\omega}\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}}\right) \tag{20}$$

which are the relative WFs corresponding to eigenvalues of the harmonic oscillators with $2n+1$ eigenvalues. To obtain Eq. (20), we have used the fact that every power of $\overline{q}^2+\partial_{\overline{p}}^2$ commutes with Laguerre polynomials with argument of $\overline{q}\mp i\ \partial_{\overline{p}}$ in Eq. (19), and the identity

$$\sqrt{\frac{\overline{\omega}}{2}}\left(\overline{q}\mp i\partial_{\overline{p}}\right)L_n^{1/2}\left[\frac{\overline{\omega}}{2}\left(\overline{q}\mp i\ \partial_{\overline{p}}\right)^2\right]=\frac{(-1)^n}{2^{2n+1}n!}H_{2n+1}\left[\sqrt{\frac{\overline{\omega}}{2}}\left(\overline{q}\mp i\ \partial_{\overline{p}}\right)\right], \tag{21}$$

and we have derived a new identity between Hermite and Laguerre polynomials in the form of

$$H_n(\overline{u}+\frac{i}{2}\partial_{\overline{v}})H_n(\overline{u}-\frac{i}{2}\partial_{\overline{v}})e^{-\overline{v}^2}=(-1)^n2^nn!L_n\left[2\left(\overline{u}^2+\overline{v}^2\right)\right]e^{-\overline{v}^2}. \tag{22}$$

The proof of Eq. (22) can easily be done by using standard relations among these polynomials. In case of $\alpha=0$, we proceed as before by formally using Eqs. (21) and (22) to give

$$\widetilde{W}_{n0}\left(\overline{q},\overline{p}\right)\ =\ \frac{\sqrt{\pi}\,(2n)!}{n!\ 2^{2n}\ \Gamma\left(n+1/2\right)}\exp\left(-\frac{\overline{\omega}\overline{q}^2}{2}-\frac{\overline{p}^2}{2\overline{\omega}}\right)L_{2n}\left(\overline{\omega}\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}}\right) \tag{23}$$

which are the relative WFs corresponding to eigenvalues of the harmonic oscillators with $2n$ eigenvalues, again with $n=0,1,2,3,\ldots$. A more general expression for these two cases can be found by noticing that arrangement of the coefficients in Eqs. (20) and (23) leads to the pair of equations

$$\widetilde{W}_n\left(\overline{q},\overline{p}\right)=\begin{cases}-\exp\left(-\frac{\overline{\omega}\overline{q}^2}{2}-\frac{\overline{p}^2}{2\overline{\omega}}\right)\ L_{2n+1}\left(\overline{\omega}\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}}\right),\\ +\exp\left(-\frac{\overline{\omega}\overline{q}^2}{2}-\frac{\overline{p}^2}{2\overline{\omega}}\right)\ L_{2n}\left(\overline{\omega}\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}}\right),\end{cases}$$

or they may be combined into the form of

$$\widetilde{W}_n\left(\overline{q},\overline{p}\right)=(-1)^n\exp\left(-\frac{\overline{\omega}\overline{q}^2}{2}-\frac{\overline{p}^2}{2\overline{\omega}}\right)\ L_n\left(\overline{\omega}\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}}\right). \tag{24}$$

The WFs constructed from the products of Eq. (24) with the Eq. (18) define the WFs of two-noninteracting particles confined in a harmonic well potential in one dimension, or alternatively, they define WFs of a particle in a harmonic potential in two-space dimensions. When $g\neq 0$, which indicates that α would be greater than 1, then the total WF becomes

$$\begin{aligned}\widetilde{W}_{l,n}\left(\overline{q},\overline{p};\overline{Q},\overline{P}\right)&=(-1)^l\ \exp\left[-2\overline{Q}^2-2\overline{P}^2\right]\\ &\times\ L_l\left(4\overline{Q}^2+4\overline{P}^2\right)\frac{n!}{\Gamma\left(n+\alpha+1/2\right)}e^{-\overline{\omega}\overline{q}^2/2}\\ &\times L_n^{\alpha-1/2}\left[\frac{\overline{\omega}}{2}\left(\overline{q}-i\ \partial_{\overline{p}}\right)^2\right]L_n^{\alpha-1/2}\left[\frac{\overline{\omega}}{2}\left(\overline{q}+i\ \partial_{\overline{p}}\right)^2\right]\\ &\times\left[\frac{\overline{\omega}}{2}\left(\overline{q}^2+\partial_{\overline{p}}^2\right)\right]^\alpha\ e^{-\overline{p}^2/2\overline{\omega}}.\end{aligned} \tag{25}$$

Finally, by these considerations, we comment on $\widetilde{W}_n(\overline{q},\overline{p})$ given by Eq. (24), rather than Eq. (25), to better visualize the phase space behaviors of WFs presented in Figs. 1-2., i.e., how the

localization in $\overline{q}$ happens when the strength of spatial confinement $\overline{\omega}$ is increased. The use of the asymptotic expansion of the Laguerre polynomials for large order [38] yields Eq. (24) to take form

$$\begin{aligned} \widetilde{W}_n(\overline{q},\overline{p}) &\simeq \frac{(-1)^n}{\sqrt{\pi}}\left[(n+\frac{1}{2})(\overline{\omega}\,\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}})\right]^{-1/4} \\ &\times\ cos\left\{2\left[(n+\frac{1}{2})(\overline{\omega}\,\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}})\right]^{1/2}-\frac{\pi}{4}\right\}, \end{aligned} \tag{26}$$

from which one can easily find out where the WF is zero and where it takes negative values. For instance, Eq. (26) has zeros when

$$2\left[(n+\frac{1}{2})(\overline{\omega}\,\overline{q}^2+\frac{\overline{p}^2}{\overline{\omega}})\right]^{1/2}-\frac{\pi}{4}=(k-\frac{1}{2})\pi, \qquad k=0,\pm1,\pm2,\ldots, \tag{27}$$

which clarifies the above comment that the localization in $\overline{q}$ appears as $\overline{\omega}$ increases. Namely, the left hand side of Eq. (27) can be rearranged, in dimensional units as usual, to give

$$\frac{2\mathcal{H}(q,p)}{\omega}=\frac{\pi^2}{4(n+\frac{1}{2})}(k-\frac{1}{4})^2\hbar. \tag{28}$$

In fact, this last expression is called the symplectic area enclosed by an ellipse whose boundary is given by $\mathcal{H}(q,p)=E=p^2/(2m)+\omega^2q^2/2$ and its minimum value is determined by the Gromov's non-squeezing theorem, i.e., $2\mathcal{H}(q,p)/\omega\geq\hbar$ [39]. Therefore, the projections of WFs onto (q,p) plane are elliptic energy shells whose eccentricity is given by $e=\sqrt{1-(m\omega)^2}$, and they are circles with the frequency $\omega=1/m$.

In this paper, we introduced and solved the WFs of one-dimensional two particle Calogero-Sutherland system in which the particles obeying the Boltzman statistics interact mutually by the sum of quadratic and inversely quadratic pair potentials, and they are confined in an external harmonic potential as well. It is obvious that the technique introduced here can easily be extended to find explicit analytical expressions for WFs of 3-and N-body counterpart of the problem. Namely, by using Jacobi coordinates, after separating the center-of-mass coordinate, one can easily construct the remaining part of WF with $N-1$ relative coordinates. Furthermore, due to the fact that, with particular choices of the coupling constant g, the radical equation of three dimensional isotropic oscillator and of hydrogen-like atom in both spherical and parabolic coordinates, one dimensional three body problem and the s-state of Morse potential[22] are all reduced to Calogero-Sutherland system, the results obtained here unify inherently the WFs of these quantum mechanical problems.

As a final remark, we should point out that the attractive interaction, i.e., $0>g\geq-\hbar^2/8\mu$, is also present in the CSM. Therefore, one can easily compare the phase space behaviors of WFs of two different regimes as well. In particular, it should be noted that a particular choice of g, when $0>g\geq-\hbar^2/8\mu$, yields one dimensional band problem solved by Scarf[20, 21]. This serves as a model for one dimensional dot arrays, as also indicated in the Introduction section.

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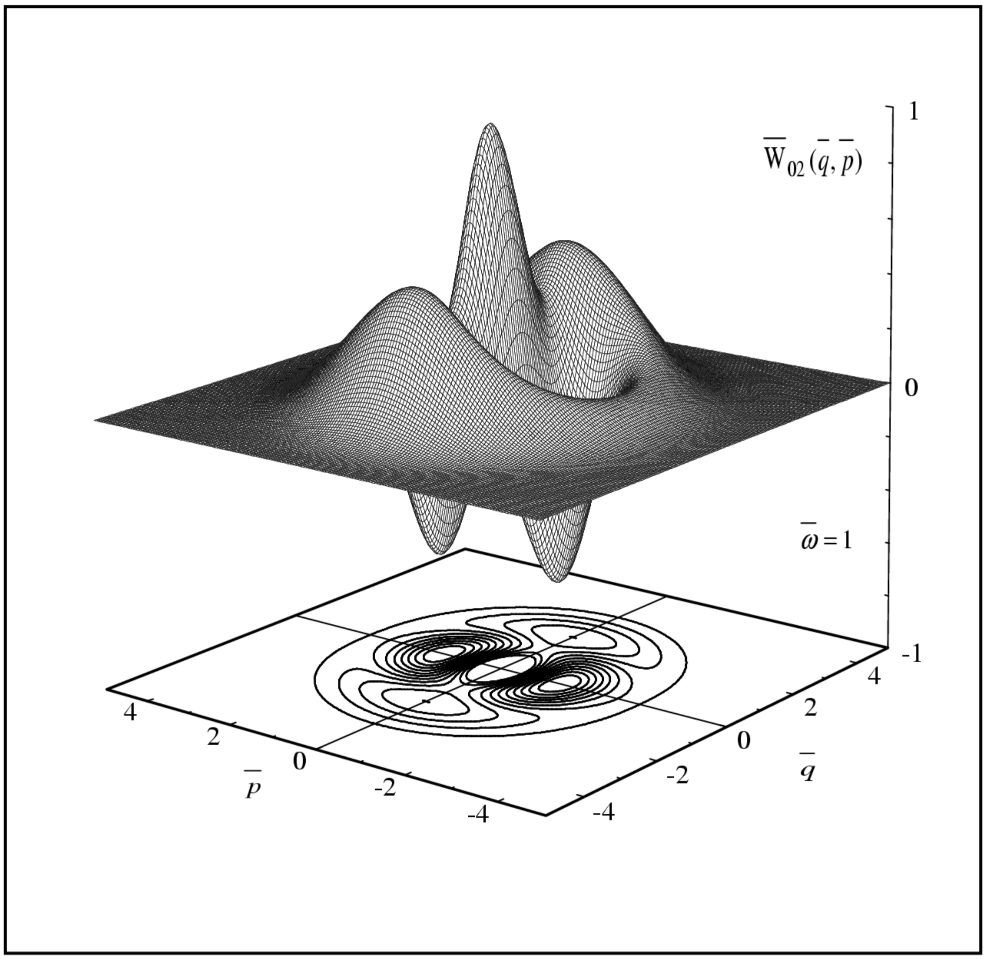
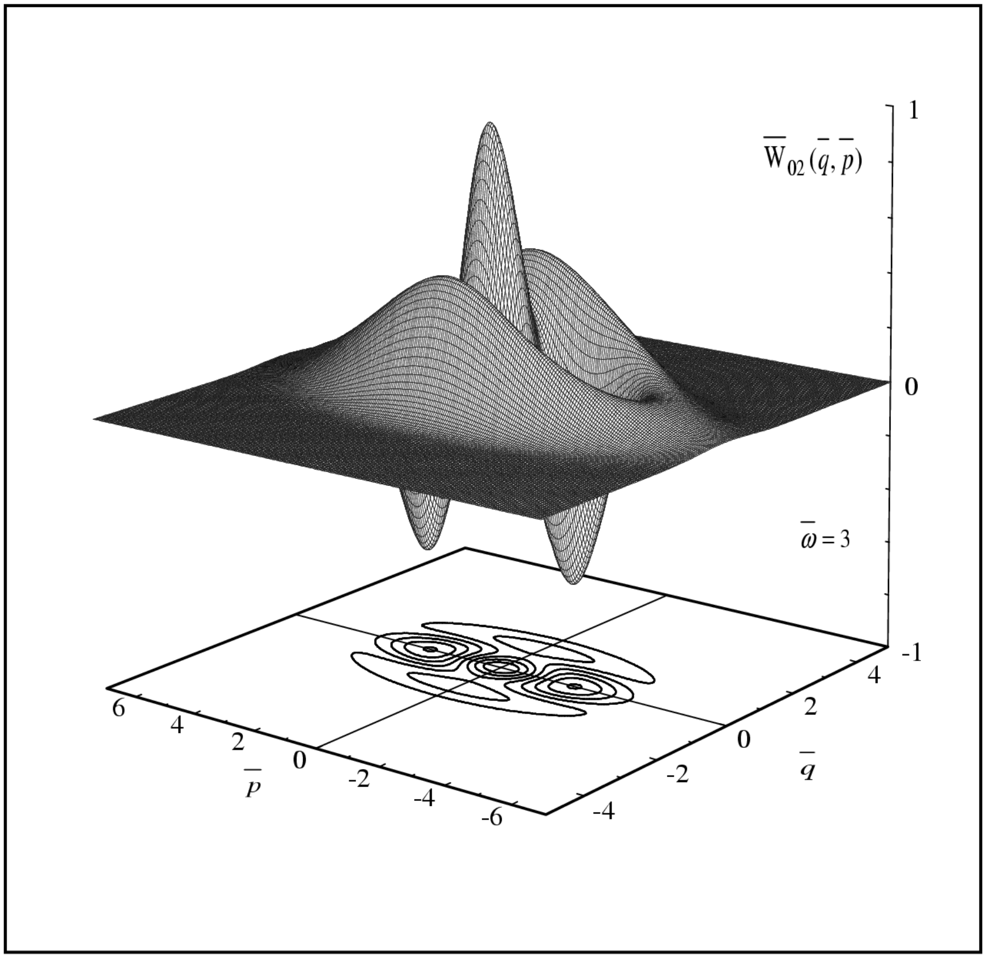
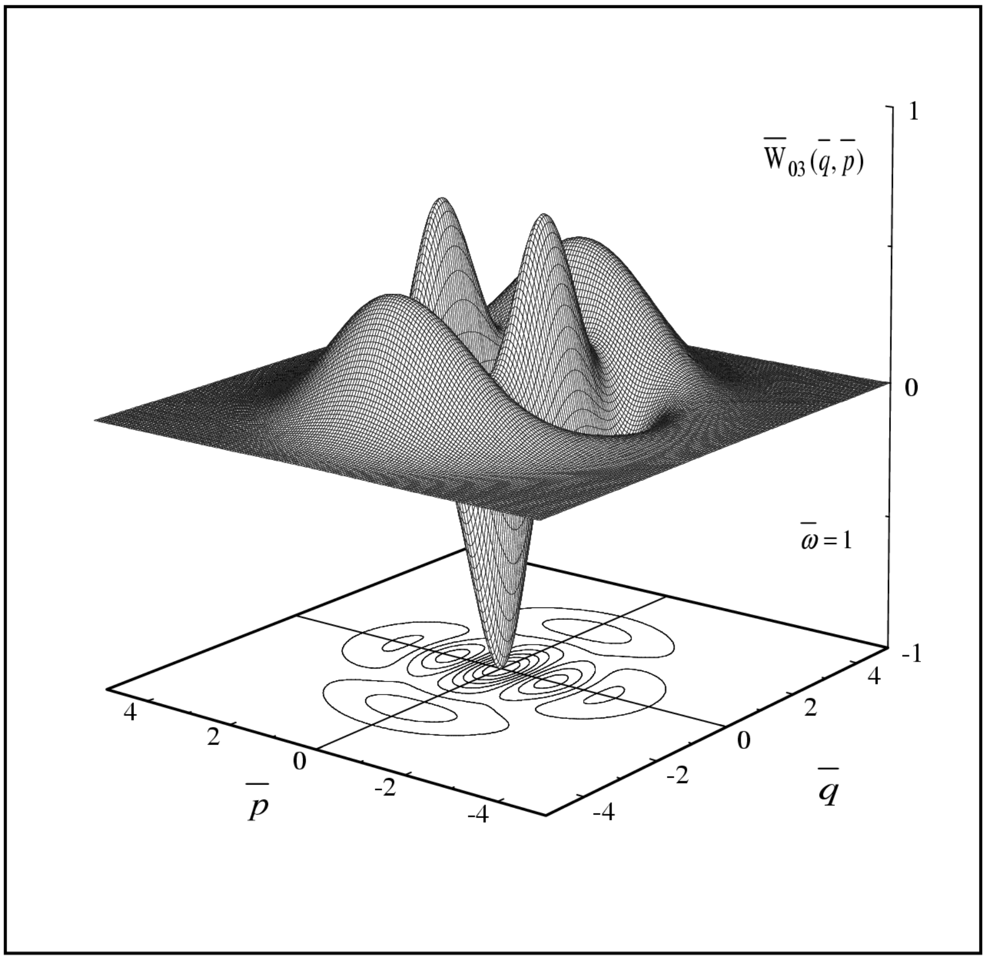


(a)

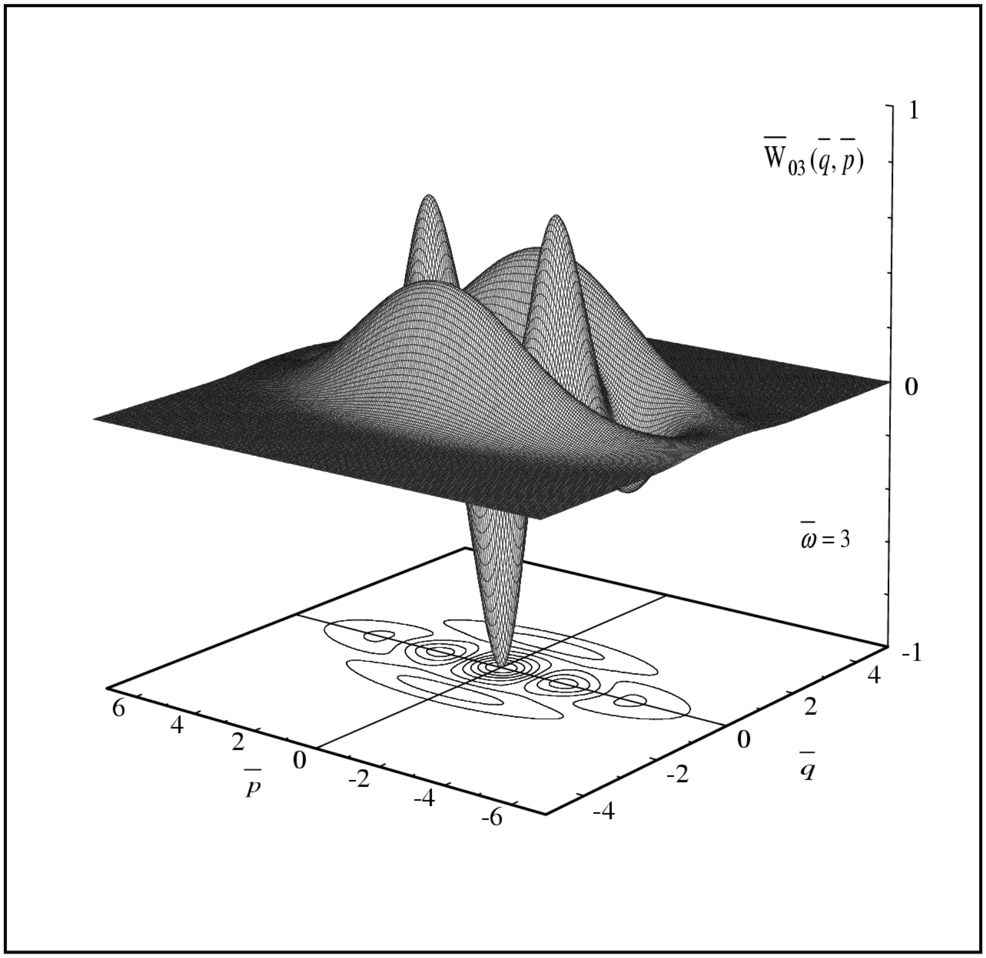


(b)

Figure 1: Three dimensional plots of WF $\widetilde{W}_{02}$ for the relative motion (Eq. (19)) as a function of dimensionless position $\overline{q} = q/l$ and momentum $\overline{p} = lp/\hbar$ for dimensionless frequency (a) $\overline{\omega} = \omega/\omega_\bullet = 1$ and (b) $\overline{\omega} = 3$.



(a)



(b)

Figure 2: Three dimensional plots of WF $\widetilde{W}_{03}$ for the relative motion (Eq. (19)) as a function of dimensionless position $\overline{q} = q/l$ and momentum $\overline{p} = lp/\hbar$ for dimensionless frequency (a) $\overline{\omega} = \omega/\omega_\bullet = 1$ and (b) $\overline{\omega} = 3$.